SOME ASYMPTOTIC BESSEL FUNCTION RATIOS

BY

JAMES E. KIEFER AND GEORGE H. WEISS

ABSTRACT

We derive leading terms in the expansion of ratios of the form $B_{n+\alpha}(n\beta)/B_n(n\beta)$ for large n, where $B_n(x)$ is any one of the Bessel functions $J_n(x)$, $Y_n(x)$, $H_n(x)$, $I_n(x)$ and $K_n(x)$.

In a recent investigation of the effects of gradients in gel permeation chromatography [1], we required the value of $\lim_{n\to\infty} I_{n+\frac{1}{2}}(n\beta)/I_n(n\beta)$ where $I_m(x)$ is the Bessel function of order m of imaginary argument. An investigation of relevant literature showed that although results of the form $\lim_{n\to\infty} B_{n+\alpha}(n\beta)/B_n(n\beta)$, where $B_m(x)$ is any one the Bessel functions $J_m(x)$, $Y_m(x)$, $H_m(x)$, $I_m(x)$, and $K_m(x)$, were implicit in the work of earlier investigators, they have not been given explicitly except for the cases $I_{n\pm 1}(n\beta)/I_n(n\beta)$ derived by Montroll [2]. Accordingly it is the purpose of this note to give the first two terms in the asymptotic expansion of these Bessel function ratios.

To begin with, we note the existence of asymptotic series for large ν, of which the following, due to Debye [3], is typical:

$$J_\nu(\nu \operatorname{sech}\lambda) \sim (2\pi\nu\tanh\lambda)^{-\frac{1}{2}} e^{\nu(\tanh\lambda-\lambda)} \sum_{r=0}^{\infty} u_r(\coth\lambda)\nu^{-r} \tag{1}$$

where the $u_k(t)$ are polynomials generated from

$$\begin{cases} u_0(t) = 1 \\ u_{r+1}(t) = \frac{1}{2}t^2(1-t^2)u_r'(t) + \dfrac{1}{8}\displaystyle\int_0^t (1-5\tau^2)u_r(\tau)\,d\tau, \qquad r \geqq 0. \end{cases} \tag{2}$$

In order to generate the value of $J_{n+\alpha}(n\beta)/J_n(n\beta)$ for large β from Eq. (1) we set $\nu = n+\alpha$, and $\operatorname{sech}\lambda = n\beta/(n+\alpha)$ in the numerator and $\nu = n$, $\operatorname{sech}\lambda = \beta$ in the

Received March 1, 1972

denominator. It is then straightforward but tedious to verify that for $\beta < 1$, we have

$$(3)\quad \begin{cases} J_{n+\alpha}(n\beta)/J_n(n\beta) = \left(\dfrac{\beta}{1+\sqrt{1-\beta^2}}\right)^\alpha \left[1 - \dfrac{1+\alpha\sqrt{1-\beta^2}}{2(1-\beta^2)}\dfrac{\alpha}{n} + O\left(\dfrac{1}{n^2}\right)\right] \\ Y_{n+\alpha}(n\beta)/Y_n(n\beta) = \left(\dfrac{1+\sqrt{1-\beta^2}}{\beta}\right)^\alpha \left[1 - \dfrac{1-\alpha\sqrt{1-\beta^2}}{2(1-\beta^2)}\dfrac{\alpha}{n} + O\left(\dfrac{1}{n^2}\right)\right] \\ H_{n+\alpha}(n\beta)/H_n(n\beta) = \left(\dfrac{\beta}{1+\sqrt{1-\beta^2}}\right)^\alpha \left[1 - \dfrac{1+\alpha\sqrt{1-\beta^2}}{2(1-\beta^2)}\dfrac{\alpha}{n} + O\left(\dfrac{1}{n^2}\right)\right] \end{cases}$$

In all of these formulae it is assumed that $n(1-\beta^2) \gg 1$. When $\beta = 1$, we can use expansions first given by Meissels [4], to prove the results

$$(4)\quad \begin{cases} J_{n+\alpha}(n)/J_n(n) = 1 - \alpha r u + \left(\dfrac{\alpha^3}{18} - \dfrac{\alpha}{45}\right)u^3 + \dfrac{r}{36}(\alpha^2 - \alpha^4)u^4 + O(u^6) \\ Y_{n+\alpha}(n)/Y_n(n) = 1 + \alpha r u + \left(\dfrac{\alpha^3}{18} - \dfrac{\alpha}{45}\right)u^3 - \dfrac{r}{36}(\alpha^2 - \alpha^4)u^4 + O(u^6) \\ H_{n+\alpha}(n)/H_n(n) = 1 - \omega\alpha r u + \left(\dfrac{\alpha^3}{18} - \dfrac{\alpha}{45}\right)u^3 + \dfrac{\omega r}{36}(\alpha^2 - \alpha^4)u^4 + O(u^6) \end{cases}$$

in which we have used the notation

$$(5)\qquad r = \Gamma(2/3)/\Gamma(1/3),\ u = (6/n)^{1/3},\ \omega = \tfrac{1}{2}(-1 + i\sqrt{3}).$$

When $\beta > 1$, the ratios for J and Y do not have limiting values, but the ratio for the H functions does, and the result can be expressed as

$$(6)\quad H_{n+\alpha}(n\beta)/H_n(n\beta) = \left(\frac{\beta}{1+i\sqrt{\beta^2-1}}\right)^\alpha \left[1 + \frac{1+\alpha\sqrt{\beta^2-1}}{2(\beta^2-1)}\frac{\alpha}{n} + O\left(\frac{1}{n^2}\right)\right].$$

The Bessel functions of imaginary argument can be handled in the same way and with no restriction on the value of β we have

$$(7a)\quad I_{n+\alpha}(n\beta)/I_n(n\beta) = \left(\frac{\beta}{1+\sqrt{\beta^2+1}}\right)^\alpha \left[1 - \frac{1+\alpha\sqrt{\beta^2+1}}{2(1+\beta^2)}\frac{\alpha}{n} + O\left(\frac{1}{n^2}\right)\right]$$

$$(7b)\quad K_{n+\alpha}(n\beta)/K_n(n\beta) = \left(\frac{1+\sqrt{\beta^2+1}}{\beta}\right)^\alpha \left[1 - \frac{1-\alpha\sqrt{\beta^2+1}}{2(1+\beta^2)}\frac{\alpha}{n} + O\left(\frac{1}{n^2}\right)\right]$$

The result in Eq. (7a) agrees with that given earlier by Montroll [2] for $\alpha = \pm 1$. Further results along these lines can be derived for other types of Bessel functions by the same means.

Acknowledgement

We are grateful to Professor F. J. Olver for a useful discussion.

References

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Physical Sciences Laboratory, Division of Computer Research & Technology
National Institutes of Health
Bethesda, Maryland